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The Energy–Momentum Tensor in Yang–Mills Field Theory and Its Uniqueness

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The uniqueness of the energy-momentum tensor in Yang-Mills field theory is established under general conditions.

1. INTRODUCTION

Let $P(M, G, \Pi)$ be a principle fiber bundle with base space M, total space P and structural group G. Let $n = \dim M$, $r = \dim G$. For α, β non-negative integers we define $V := T^{\alpha}_{\beta}(LG)$, the space of α-contravariant, β-covariant tensors on the Lie algebra LG of G and ρ: $G \to GL(V)$ by

$$\rho := \mathrm{Ad} \otimes \cdots \otimes \widetilde{\mathrm{Ad}} \otimes \mathrm{Ad} \otimes \cdots \otimes \widetilde{\mathrm{Ad}}, \tag{1}$$

where

$$\widetilde{\mathrm{Ad}}(a)(\eta)(X_e) := \eta[\mathrm{Ad}(a)(X_e)] \tag{2}$$

and Ad is the adjoint representation of G. Let z be the local chart around e in G given by exp.

A *gauge field* is a connection form ω on P. If U is an open set in M then a *gauge* is a pair (U, σ) where σ: $U \to P$ is a smooth section of Π. For a gauge (U, σ) let $\omega_\sigma := \sigma^* \omega$. Then ω_σ is an LG-valued 1-form defined on U. If (x, V) is a local chart in M such that $U \cap V \neq \emptyset$ then

$$\omega_\sigma = (A_i^\alpha \, dx^i) e_\alpha$$

(latin letters run from 1 to n, Greek letters run from 1 to r, and we use the summation convention). The A_i^α are called the *gauge potentials* of ω associated to (U, σ), (x, V), and e_α.

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If (U, α) and (U', σ') are two gauges such that $\sigma(m) = \sigma'(m)$ for some m in $U \cap U'$ then there is a smooth function ψ: $U \cap U' \to G$ such that $\sigma \cdot \psi = \sigma'$ in $U \cap U'$. It is well known that

$$A_i'^{\alpha} = Ad^{\alpha}_{\beta} \circ \psi^{-1} A_i^{\beta} + l^{\alpha}_{\beta} \circ \psi \frac{\partial \psi^{\beta}}{\partial x^i} \tag{3}$$

where $\psi^{\beta} := z^{\beta} \circ \psi$; $l^{\alpha}_{\beta}\, dz^{\beta}$ are the left invariant 1-forms generated by the dual basis of e_{α}, and $\mathrm{Ad}(a) e_{\alpha} = \mathrm{Ad}^{\beta}_{\alpha}(a) e_{\beta}$.

We say that T is a *gauge tensor field of type* $(V; r, s, w)$ if it gives for every gauge (U, σ) a V-valued relative tensor field T_{σ} of type (r, s, w) defined over U. We say that T is a gauge tensor field of type $(\rho; r, s, w)$ if furthermore

$$T_{\sigma'} = \rho(\psi^{-1}) T_{\sigma} \qquad \text{in } U \cap U' \tag{4}$$

where ρ is given by (1).

The coefficients of the curvature form, defined as

$$F^{\alpha}_{ij} := A^{\alpha}_{j,i} - A^{\alpha}_{i,j} + C^{\alpha}_{\beta\gamma} A^{\beta}_i A^{\gamma}_j \tag{5}$$

where $C^{\alpha}_{\beta\gamma}$ are the structure constants associated to e_{α}, are the components of a gauge tensor F of type (Ad; 0, 2, 0).

If we have a Lorentz metric g_{ij} on M, the gauge-covariant derivative of F is defined as

$$F^{\alpha}_{ij\|h} := F^{\alpha}_{ij,h} - F^{\alpha}_{kj} \Gamma^k_{ih} - F^{\alpha}_{ik} \Gamma^k_{jh} + F^{\gamma}_{ij} C^{\alpha}_{\beta\gamma} A^{\beta}_h \tag{6}$$

where Γ^k_{jh} are the Christoffel symbols.

In Yang–Mills field theory the role played by the energy-momentum tensor

$$T^{ij} = \sqrt{g}\, C_{\alpha\beta} (F^{\alpha ir} F^{\beta j}_{r} - \tfrac{1}{4} g^{ij} F^{\alpha}_{hs} F^{\beta hs}) \tag{7}$$

where $C_{\alpha\beta} := C^{\gamma}_{\alpha\varepsilon} C^{\varepsilon}_{\beta\gamma}$ are the coefficients of the Cartan–Killing form, is well known. It has the following properties:

(a) $T^{ij} = T^{ji}$ (8)

(b) Whenever the Yang–Mills equations

$$C_{\alpha\beta} F^{\alpha ij}{}_{\|j} = 0 \tag{9}$$

are satisfied, the divergence of T^{ij}, i.e., $T^{ij}_{\|j}$, vanishes by virtue of the identity (Rund, 1982)

$$T^{ij}{}_{\|j} = \sqrt{g}\; C_{\alpha\beta} F^{\beta ir} F^{\alpha j}{}_{r\|j} \tag{10}$$

The purpose of this paper is to establish that T^{ij} is essentially the unique solution to the following problem. To find all gauge tensors B^{ij} for which (i) B^{ij} is a concomitant of g_{ab}, A^{α}_{a} and $A^{\alpha}_{a,b}$, i.e.,

$$B^{ij}=B^{ij}(g_{ab};\,A^{\alpha}_{a};\,A^{\alpha}_{a,b}) \tag{11}$$

(ii) $B^{ij}=B^{ji}$ (12)

(iii) $B^{ij}{}_{\|i}$ vanishes whenever (9) is valid in the sense that

$$B^{ij}{}_{\|i}=C_{\alpha\beta}H^{\beta jr}F^{\alpha i}{}_{r\|i} \tag{13}$$

where $H^{\beta jr}=H^{\beta jr}(g_{hk};\,A^{\alpha}_{h};\,A^{\alpha}_{h,k})$ is a gauge tensor.

If $H^{jr}_{\alpha}=C_{\alpha\beta}F^{\beta jr}$, then (13) means

$$B^{ij}{}_{\|i}=H^{jr}_{\alpha}F^{\alpha i}{}_{r\|i} \tag{14}$$

This is a natural extension of the same problem in electromagnetism treated by Lovelock (1974).

2. THE UNIQUENESS OF THE ENERGY–MOMENTUM TENSOR

Since B^{ij} is a gauge tensor field, then from the replacement theorem (Horndeski, 1981) we have

$$B^{ij}(g_{hk};\,A^{\alpha}_{h};\,A^{\alpha}_{h,k})=B^{ij}(g_{hk};\,0;\,-\tfrac{1}{2}F^{\alpha}_{hk})=B^{ij}_{1}(g_{hk};\,F^{\alpha}_{hk}) \tag{15}$$

and B^{ij}_{1} is also a gauge tensor field. By (15) we see that

$$\frac{\partial B^{ij}}{\partial A^{\alpha}_{h,k}}+\frac{\partial B^{ij}}{\partial A^{\alpha}_{k,h}}=0 \tag{16}$$

Since, for a fixed gauge, B^{ij} is a tensorial concomitant it must satisfy certain invariance identities (Rund, 1966). One set of them is a consequence of (16) and the other set is

$$2\frac{\partial B^{ij}}{\partial g_{hb}}g_{ha}-\frac{\partial B^{ij}}{\partial A^{\alpha}_{h,b}}F^{\alpha}_{ha}=-\delta^{i}_{a}B^{bj}-\delta^{j}_{a}B^{ib} \tag{17}$$

The identity (14), written out in full, is

$$\frac{\partial B^{ij}}{\partial g_{hk}}g_{hk,i}+\frac{\partial B^{ij}}{\partial A^{\alpha}_{h}}A^{\alpha}_{h,i}+\frac{\partial B^{ij}}{\partial A^{\alpha}_{h,k}}A^{\alpha}_{h,ki}+\Gamma^{i}_{si}B^{sj}+\Gamma^{j}_{si}B^{is}$$
$$=H^{jr}_{\alpha}g^{is}[F^{\alpha}_{sr,i}-F^{\alpha}_{kr}\Gamma^{k}_{si}-F^{\alpha}_{sk}\Gamma^{k}_{ri}+F^{\gamma}_{sr}C^{\alpha}_{\beta\gamma}A^{\beta}_{i}] \tag{18}$$

Differentiating with respect to $g_{ab,c}$ gives

$$2\frac{\partial B^{cj}}{\partial g_{ab}}+g^{ab}B^{cj}+g^{jb}B^{ca}+g^{ja}B^{cb}-g^{jc}B^{ab}$$
$$=H^{jr}_{\alpha}[-g^{ca}g^{kb}F^{\alpha}_{kr}-g^{cb}g^{ka}F^{\alpha}_{kr}+g^{ab}g^{kc}F^{\alpha}_{kr}]$$
$$-H^{ja}_{\alpha}g^{cs}g^{kb}F^{\alpha}_{sk}-H^{jb}_{\alpha}g^{cs}g^{ka}F^{\alpha}_{sk} \tag{19}$$

while differentiation of (18) with respect to $A^{\alpha}_{h,ki}$ gives

$$\frac{\partial B^{ij}}{\partial A^{\alpha}_{h,k}}+\frac{\partial B^{kj}}{\partial A^{\alpha}_{h,i}}=-H^{jk}_{\alpha}g^{ih}-H^{ji}_{\alpha}g^{kh}+2H^{jh}_{\alpha}g^{ih} \tag{20}$$

Interchanging j with k and j with i on (20) we obtain

$$\frac{\partial B^{ik}}{\partial A^{\alpha}_{h,j}}+\frac{\partial B^{jk}}{\partial A^{\alpha}_{h,i}}=-H^{kj}_{\alpha}g^{ih}-H^{ki}_{\alpha}g^{jh}+2H^{kh}_{\alpha}g^{ij} \tag{21}$$

and

$$\frac{\partial B^{ji}}{\partial A^{\alpha}_{h,k}}+\frac{\partial B^{ki}}{\partial A^{\alpha}_{h,j}}=-H^{ik}_{\alpha}g^{jh}-H^{ij}_{\alpha}g^{kh}-H^{ij}_{\alpha}g^{kh}+2H^{ih}_{\alpha}g^{jk} \tag{22}$$

Adding (20) and (21) and subtracting (22) we have

$$2\frac{\partial B^{ki}}{\partial A^{\alpha}_{h,i}}=-H^{jk}_{\alpha}g^{ih}-H^{ji}_{\alpha}g^{kh}+2H^{jh}_{\alpha}g^{ik}-H^{kj}_{\alpha}g^{ih}-H^{ki}_{\alpha}g^{jh}$$
$$+2H^{kh}_{\alpha}g^{ij}+H^{ik}_{\alpha}g^{jh}+H^{ij}_{\alpha}g^{kh}-2H^{ih}_{\alpha}g^{jk} \tag{23}$$

Interchanging h with i in (23), adding it to (23), and contracting with g_{jh} we deduce

$$(n-2)(H^{ki}_{\alpha}+H^{ik}_{\alpha})+2g^{ki}H^{jh}_{\alpha}g_{jh}=0 \tag{24}$$

Contracting (24) with g_{ki} we see that

$$4(n-1)H^{ki}_{\alpha}g_{ki}=0 \tag{25}$$

From (24) and (25) we deduce that if $n>2$ then

$$H^{ki}_{\alpha}=-H^{ik}_{\alpha} \tag{26}$$

and so (23) reduces to

$$\frac{\partial B^{kj}}{\partial A^{\alpha}_{h,i}}=H^{ij}_{\alpha}g^{kh}+H^{jh}_{\alpha}g^{ik}+H^{ik}_{\alpha}g^{jh}+H^{kh}_{\alpha}g^{ij}+H^{hi}_{\alpha}g^{jk} \tag{27}$$

Contracting (27) with g_{kh} we obtain

$$H^{ij}_{\alpha}=\frac{1}{n-1}\frac{\partial B^{kj}}{\partial A^{\alpha}_{h,i}}g_{kh}$$

and then it is easy to deduce that

$$H^{ij}_{\alpha}(g_{hk};\, A^{\alpha}_{h};\, A^{\alpha}_{h,k})=H^{ij}_{\alpha}(g_{hk};\, 0;\, -\tfrac{1}{2}F^{\alpha}_{hk})$$
$$=H_{1}{}^{ij}_{\alpha}(g_{hk};\, F^{\alpha}_{hk}) \tag{28}$$

Differentiating (27) with respect to $A^{\beta}_{r,s}$ we have

$$\frac{\partial^2 B^{kj}}{\partial A^{\alpha}_{h,i}\,\partial A^{\beta}_{r,s}} = H^{ij;\,r,s}_{\alpha\ \beta} g^{kh} + H^{jh;\,r,s}_{\alpha\ \beta} h^{ik} + H^{ik;\,r,s}_{\alpha\ \beta} g^{jh} + H^{kh;\,r,s}_{\alpha\ \beta} g^{ij} + H^{hi;\,r,s}_{\alpha\ \beta} g^{jk} \tag{29}$$

From (29) we deduce

$$\begin{aligned} H^{ij;\,r,s}_{\alpha\ \beta} g^{kh} + H^{jh;\,r,s}_{\alpha\ \beta} g^{ik} + H^{ik;\,r,s}_{\alpha\ \beta} g^{jh} + H^{kh;\,r,s}_{\alpha\ \beta} g^{ij} + H^{hi;\,r,s}_{\alpha\ \beta} g^{jk} \\ = H^{sj;\,h,i}_{\beta\ \alpha} g^{kr} + H^{jr;\,h,i}_{\beta\ \alpha} g^{sk} + H^{sk;\,h,i}_{\beta\ \alpha} g^{jr} \\ + H^{kr;\,h,i}_{\beta\ \alpha} g^{sj} + H^{rs;\,h,i}_{\beta\ \alpha} g^{jk} \end{aligned} \tag{30}$$

Contracting (30) with g_{kh} we obtain

$$\begin{aligned} (n-1) H^{ij;\,r,s}_{\alpha\ \beta} = H^{sj;\,r,i}_{\beta\ \alpha} + H^{jr;\,s,i}_{\beta\ \alpha} + H^{sk;\,h,i}_{\beta\ \alpha} g_{kh} g^{jr} \\ + H^{kr;\,h,i}_{\beta\ \alpha} g_{kh} g^{sj} + H^{rs;\,j,i}_{\beta\ \alpha} \end{aligned} \tag{31}$$

Contracting (31) with g_{is} it is easy to deduce that

$$H^{ij;\,r,s}_{\alpha\ \beta} g_{is} = \lambda_{\alpha\beta} g^{jr} \tag{32}$$

where

$$\lambda_{\alpha\beta} = \frac{1}{n} H^{sk;\,h,i}_{\beta\ \alpha} g_{is} g_{kh} = \lambda_{\beta\alpha} \tag{33}$$

Replacing (32) in (31) yields

$$\begin{aligned} (n-1) H^{ij;\,r,s}_{\alpha\ \beta} = H^{sj;\,r,i}_{\beta\ \alpha} + H^{jr;\,s,i}_{\beta\ \alpha} + H^{rs;\,j,i}_{\beta\ \alpha} \\ + \lambda_{\alpha\beta}(g^{is}g^{jr} - g^{ir}g^{sj}) \end{aligned} \tag{34}$$

The left-hand side of (34) is invariant when we interchange i with j and r with s by virtue of (26) and (28). Then we deduce

$$H^{sj;\,r,i}_{\beta\ \alpha} + H^{jr;\,s,i}_{\beta\ \alpha} = H^{ri;\,s,j}_{\beta\ \alpha} + H^{is;\,r,j}_{\beta\ \alpha} \tag{35}$$

Interchanging j with s in (35), interchanging j with r in (35), and adding the three equations we obtain

$$H^{sr;\,j,i}_{\beta\ \alpha} = H^{ji;\,s,r}_{\beta\ \alpha} \tag{36}$$

Interchanging r with j in (34) and adding it to (34) we have

$$H^{ij;\,r,s}_{\alpha\ \beta} + H^{ir;\,j,s}_{\alpha\ \beta} = \frac{\lambda_{\alpha\beta}}{n-1}[2g^{is}g^{rj} - g^{ir}g^{sj} - g^{ij} - g^{ij}g^{sr}] \tag{37}$$

Using (37) we can reduce (34) to

$$(n-1)H_{\alpha\ \beta}^{ij;r,s} = 3H_{\beta\ \alpha}^{sj;r,i} + \lambda_{\alpha\beta}\left[\frac{3}{n-1}g^{ij}g^{rs} - g^{ir}g^{sj} + \left(1-\frac{3}{n-1}\right)g^{is}g^{jr}\right] \tag{38}$$

From (38) it is easy to obtain, for $n>2$ and $n\neq 4$ that

$$H_{\beta\ \alpha}^{sj;r,i} = \frac{\lambda_{\alpha\beta}}{n-1}(g^{is}g^{rj} - g^{ij}g^{rs}) \tag{39}$$

Differentiating (39) with respect to $A_{h,k}^{\gamma}$ yields

$$H_{\beta\ \alpha\ \gamma}^{sj;r,i;h,k} = \frac{1}{3n-1}\frac{\partial\lambda_{\alpha\beta}}{\partial A_{h,k}^{\gamma}}(g^{is}g^{rj} - g^{ij}g^{rs}) \tag{40}$$

Since the left-hand side of (40) is invariant when we interchange h with r, k with i, and α with γ, then

$$\frac{\partial\lambda_{\alpha\beta}}{\partial A_{h,k}^{\gamma}}(g^{is}g^{rj} - g^{ij}g^{rs}) = \frac{\partial\lambda_{\gamma\beta}}{\partial A_{r,i}^{\alpha}}(g^{ks}g^{hj} - g^{kj}g^{hs}) \tag{41}$$

Contracting (41) with $g_{is}g_{rj}$ we deduce

$$\frac{\partial\lambda_{\alpha\beta}}{\partial A_{h,k}^{\gamma}} = 0$$

and so, for a fixed gauge, $\lambda_{\alpha\beta}$ is a scalar concomitant of g_{hk}. Then (Lovelock, 1969) $\lambda_{\alpha\beta}$ is a real number for each α, β. Now using (28) and integrating (39) we have for $n>2$ and $n\neq 4$ that

$$H_{\beta}^{sj} = \frac{\lambda_{\alpha\beta}}{n-12}F^{\alpha js} \tag{42}$$

It remains to prove (42) for the case of physical interest, i.e., $n=4$. To achieve that we remark that (38) reduces to

$$3H_{\alpha\ \beta}^{ij;r,s} = 3H_{\beta\ \alpha}^{sj;r,i} + \lambda_{\alpha\beta}(g^{ij}g^{rs} - g^{ir}g^{js}) \tag{43}$$

From (37) and (43) it follows easily that

$$H_{\alpha\ \beta}^{sj;r,i} + H_{\beta\ \alpha}^{sj;r,i} = \tfrac{2}{3}\lambda_{\alpha\beta}(g^{jr}g^{si} - g^{ij}g^{rs}) \tag{44}$$

Now using (36) we have

$$H_{\alpha\ \beta\ \gamma}^{ij;r,s;h,k} = H_{\alpha\ \beta\ \gamma}^{ij;h,k;r,s} = H_{\alpha\ \gamma\ \beta}^{ij;r,s;h,k} \tag{45}$$

Differentiating (44) with respect to $A_{h,k}^{\gamma}$ we see that

$$H_{\alpha\ \beta\ \gamma}^{ij;r,s;h,k} + H_{\beta\ \alpha\ \gamma}^{ij;r,s;h,k} = \frac{2}{3}\frac{\partial\lambda_{\alpha\beta}}{\partial A_{h,k}^{\gamma}}(g^{is}g^{jr} - g^{ir}g^{js}) \tag{46}$$

By (45), the left-hand side of (46) is invariant when we interchange r with h and s with k. Then

$$\frac{\partial \lambda_{\alpha\beta}}{\partial A^{\gamma}_{h,k}}(g^{is}g^{jr}-g^{ir}g^{js})=\frac{\partial \lambda_{\alpha\beta}}{\partial A^{\gamma}_{r,s}}(g^{ik}g^{jh}-g^{ih}g^{jk}) \tag{47}$$

Contracting (47) with $g_{jr}g_{is}$ it is easy to obtain

$$\frac{\partial \lambda_{\alpha\beta}}{\partial A^{\gamma}_{h,k}}=0$$

and so again $\lambda_{\alpha\beta}$ is a real number for each α, β. Then by (46)

$$H^{ij;\,r,s;\,h,k}_{\alpha\ \beta\ \ \gamma}+H^{ij;\,r,s;\,h,k}_{\beta\ \alpha\ \ \gamma}=0 \tag{48}$$

By (26), (28), (45), and (48) we see that $H^{ij;\,r,s;\,h,k}_{\alpha\ \beta\ \ \gamma}$ is skew symmetric in all its latin indices. Since $n=4$, it follows that

$$H^{ij;\,r,s;\,h,k}_{\alpha\ \beta\ \ \gamma}=0 \tag{49}$$

Then $H^{ij;\,r,s}_{\alpha\ \beta}=H^{ij;\,r,s}_{\alpha\ \beta}(g_{hk})$, and then it is known (Lovelock, 1969) that

$$H^{ij;\,r,s}_{\alpha\ \beta}=\lambda_1 g^{ij}g^{rs}+\lambda_2 g^{ir}g^{js}+\lambda_3 g^{is}g^{jr}+\frac{4}{\sqrt{g}}\varepsilon^{ijrs}, \tag{50}$$

where ε^{ijrs} are the Levi-Cività symbols. Using a coordinate system where $(g_{ij})=\mathrm{diag}(-1,-1,-1,1)$ it follows from (26) that $\lambda_1=\lambda_4=0$ and $\lambda_2=-\lambda_3$. Now using (37) it follows that $\lambda_2=-\lambda_{\alpha\beta}/3$. Then

$$H^{ij;\,r,s}_{\alpha\ \beta}=\frac{\lambda_{\alpha\beta}}{3}(g^{is}g^{jr}-g^{ir}g^{js}) \tag{51}$$

Integrating (51) through the use of (28) it follows that

$$H^{ij}_{\alpha}=\frac{\lambda_{\alpha\beta}}{3}F^{\beta ji} \tag{52}$$

which is (42) for the case $n=4$.

Now it is standard to modify Lovelock's proof (Lovelock, 1974) to conclude the following.

Theorem. For $n>2$ the only gauge tensor which satisfies (11), (12), and (13) is

$$B^{ij}=\Lambda_{\alpha\beta}(F^{\alpha ir}F^{\beta j}_{r}-\tfrac{1}{4}g^{ij}F^{\alpha}_{hs}F^{\beta hs})+ag^{ij}$$

where a is a real number and $\Lambda_{\alpha\beta}$ satisfies

$$C^{\beta}_{\theta\gamma}\Lambda_{\beta\alpha}+C^{\beta}_{\theta\alpha}\Lambda_{\gamma\beta}=0$$

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